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A strain-based Lagrangian-history turbulence theory

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The Lagrangian-history method in turbulence theory (Kraichnan 1977) is modified such that triple moments are expanded in functional powers of the Lagrangian covariance of the symmetric rate-of-strain field instead of the Lagrangian covariance of the velocity field. The simplest approximation which results corresponds to the abridged Lagrangian-history direct-interaction approximation. It is illustrated by application to the Lagrangian properties of a random velocity field whose Eulerian values are frozen in time. Then it is formulated for isotropic Navier–Stokes turbulence. The new approximation is expected to give reduced energy transfer in the dissipation range because the rate of strain along a fluid-element trajectory is statistically stationary in stationary homogeneous turbulence while the derivatives of the Lagrangian velocity with respect to initial position tend to grow and thereby have a longer correlation time. The correlation times of these two entities play corresponding roles in the new and old approximations for energy transfer, respectively.

1. Introduction

The direct-interaction (DI), Lagrangian-history direct-interaction (LHDI) and abridged Lagrangian-history direct-interaction (ALHDI) approximations for turbulence have in common that each represents the lowest truncation of a systematic renormalized perturbation expansion of third-order (triple) moments in functional powers of the velocity covariance (Kraichnan 1977, cited hereafter as I). In the DI case, the expansion for single-time triple moments involves only purely Eulerian two-time covariances, while in the LHDI and ALHDI cases only purely Lagrangian covariances arise.

The DI approximation is of central importance because it can be represented in several ways as the exact description of a model system, thereby assuring important consistency properties (Kraichnan 1970*a*). It has given quantitatively accurate predictions for the decay of isotropic turbulence at moderate Reynolds number (Orszag & Patterson 1972), Boussinesq convection at high Prandtl number (Herring 1969) and eddy diffusion by a Gaussianly distributed velocity field (Kraichnan 1970*b*). But when applied to high Reynolds number turbulence, the DI approximation misrepresents energy transfer at high wavenumbers because the Eulerian correlation times entering the DI expression for triple moments reflect convective decorrelation effects which are irrelevant to the straining process that gives energy transfer. In two dimensions, this effect is serious even at moderate Reynolds numbers (Herring *et al.* 1974).

The spurious effects of convective decorrelation on energy transfer are removed in the LHDI and ALHDI approximations because Lagrangian rather than Eulerian correlation times are involved. In the language of I, the expansions underlying these approximations are invariant to random Galilean transformation. The LHDI and ALHDI equations have been applied with qualitative success to a broad range of turbulence problems: the inertial and dissipation ranges of high Reynolds number isotropic turbulence (Kraichnan 1966*a*), one-particle and two-particle dispersion (Kraichnan 1966*b*), Burgers-equation turbulence (Kraichnan 1968*a*), stochastic acceleration of charged particles by electric fields (Orszag 1969) and Batchelor's k^{-1} range for the spectrum of a passive scalar convected by turbulence (Kraichnan 1968*b*). In addition to giving proper exponents for various asymptotic spectrum ranges, the approximations give a wealth of information about Lagrangian correlations, acceleration statistics, and other properties not readily accessible to Eulerian treatment.

The quantitative performance, as measured by the numerical values predicted for the coefficients of the asymptotic spectral ranges, is much less satisfactory. The ALHDI predictions for Kolmogorov's constant and the dissipation-range spectrum shape are in excellent agreement with data of Grant, Stewart & Moilliet (1962). But in the cases of the Burgers equation and convection of a passive scalar, the coefficients of the k^{-2}, k^{-1} and $k^{-\frac{5}{3}}$ spectrum ranges are incorrect by factors ranging up to three. In both these cases, where it is the LHDI theory that is used, the errors correspond to overestimates of the spectral transfer associated with the straining of small scales. A comparison of the ALHDI results for the decay of isotropic turbulence with computer simulations at moderate Reynolds numbers shows a similar phenomenon (Herring & Kraichnan 1978).

In the present paper, we raise the possibility that more accurate results for spectral transfer at high wavenumbers, and some qualitative improvements as well, can be had by basing a Lagrangian-history theory on the straining field rather than directly on the velocity field. We define the straining field by

$$b_{ij}(\mathbf{x},t) = \partial u_i(\mathbf{x},t)/\partial x_j + \partial u_j(\mathbf{x},t)/\partial x_i, \tag{1.1}$$

where $u_i(\mathbf{x},t)$ is the Eulerian velocity field. If

$$\nabla . \mathbf{u}(\mathbf{x},t) = 0 \tag{1.2}$$

then

$$u_i(\mathbf{x},t) = \nabla^{-2}\,\partial \mathbf{b}_{ij}(\mathbf{x},t)/\partial x_j, \tag{1.3}$$

where ∇^{-2} is the solution operator for Poisson's equation in the given geometry.

The Eulerian moments which appear in the expansion associated with the DI approximation may be expressed indifferently in terms of either the velocity or the straining field. But this is not so for the Lagrangian covariances which appear in the LH and ALH expansions. The Lagrangian velocity covariance may be expressed in terms of the generalized velocity field $\mathbf{u}(\mathbf{x},t|s)$ defined as the velocity measured at time s (the *measuring* time) in the fluid element whose trajectory passes through the point $\mathbf{x}$ at time t (the *labelling* time). This field is related to $\mathbf{u}(\mathbf{x},t)$ by

$$\partial \mathbf{u}(\mathbf{x},t|s)/\partial t + \mathbf{u}(\mathbf{x},t) . \nabla \mathbf{u}(\mathbf{x},t|s) = 0, \quad \mathbf{u}(\mathbf{x},s|s) = \mathbf{u}(\mathbf{x},s). \tag{1.4}$$

The LH and ALH expansions of I are obtained by a perturbation treatment of (1.4) together with the equation of motion for the Eulerian field $\mathbf{u}(\mathbf{x},t)$.

In correspondence to (1.4), a generalized straining field $b_{ij}(\mathbf{x}, t|s)$ is defined by

$$\partial b_{ij}(\mathbf{x}, t|s)/\partial t + \mathbf{u}(\mathbf{x}, t) . \nabla b_{ij}(\mathbf{x}, t|s) = 0, \quad b_{ij}(\mathbf{x}, s|s) = b_{ij}(\mathbf{x}, s). \tag{1.5}$$

The point of central importance now is that, unless $t = s$,

$$b_{ij}(\mathbf{x}, t|s) \neq \partial u_i(\mathbf{x}, t|s)/\partial x_j + \partial u_j(\mathbf{x}, t|s)/\partial x_i. \tag{1.6}$$

Thus if we expand triple moments in functional powers of Lagrangian covariances of the straining field, as we shall do in the present paper, the results are not equivalent, order by order, to the LH and ALH expansions of I. The physical meaning of (1.6) is easily stated. The field $b_{ij}(\mathbf{x}, t|s)$ as a function of s for given $\mathbf{x}$ and t measures the gradient, in laboratory co-ordinates, of the Eulerian velocity field along the fluid-element trajectory. On the other hand, $\delta r_j \partial u_i(\mathbf{x}, t|s)/\partial x_j$ is the velocity difference at time s between two fluid elements whose trajectories are separated at time t by the infinitesimal distance δr_j. In general the separation at time s will not be δr_j, hence the inequality sign in (1.6). A particular important difference is that in stationary homogeneous turbulence $b_{ij}(\mathbf{x}, t|s)$ is statistically stationary as a function of s, provided that (1.2) holds so that the transformation to Lagrangian co-ordinates is measure-preserving. In general, the right-hand side of (1.6) is not stationary.

In what follows, we describe expansions and approximations developed from (1.5) in the same way that the analysis of I is developed from (1.4). The resulting approximations will be called the strain-based Lagrangian-history direct-interaction (SBLHDI) and strain-based abridged Lagrangian-history direct-interaction (SBALHDI) approximations. Our basic motivation is that the Lagrangian correlation times of the straining field may be more appropriate to description of transfer processes at small scales than the correlation times of the Lagrangian velocity field.

A question which arises here is why we use the symmetrized strain tensor instead of the unsymmetrized tensor, which is a linear combination of b_{ij} and the vorticity tensor. The answer is that we hope to construct approximations valid for two as well as three dimensions. In inviscid two-dimensional Navier–Stokes flow the vorticity of each fluid element is a constant of the motion, so that the Lagrangian correlation time of the vorticity tensor is infinite. Thus vorticity must be excluded as a component of the basic field if the final approximations, obtained by truncating the expansions at lowest order, are to give finite memory times for spectral transfer in two dimensions.

The new approximations are constructed first for the case of an Eulerian field which is random but independent of time. The analysis here is especially simple. Then the SBALHDI approximation is obtained for isotropic Navier–Stokes turbulence. The analysis is carried out for general dimensionality $D \geqslant 2$. In a companion paper (Herring & Kraichnan 1978), the SBALHDI equations for the Navier–Stokes case are integrated numerically and compared with the ALHDI approximation and with computer simulations of isotropic turbulence in two and three dimensions.

2. Some properties of the straining field

In constructing the strain-based approximations we shall consider $b_{ij}(\mathbf{x}, t)$ to be the basic field, $u_i(\mathbf{x}, t)$ playing the role of a defining vector for b_{ij}, through (1.1). Equation (1.2) implies

$$b_{ii}(\mathbf{x}, t) = 0, \quad \partial^2 b_{ij}(\mathbf{x}, t)/\partial x_i \partial x_j = 0. \tag{2.1a, b}$$

Then (1.5) yields

$$b_{ii}(\mathbf{x},t|s) = 0. \tag{2.2}$$

But there is no corresponding generalization of (2.1*b*), nor do we have $\nabla . \mathbf{u}(\mathbf{x},t|s) = 0$ unless $t = s$, except in degenerate cases. For all t and s,

$$b_{ij}(\mathbf{x},t|s) = b_{ji}(\mathbf{x},t|s). \tag{2.3}$$

In order to exhibit most simply the statistical properties of isotropic homogeneous ensembles, it is useful to let the fields obey cyclic boundary conditions in a cubical box of side L and let $L \to \infty$. The Fourier-transformed fields are then defined by

$$u_i(\mathbf{x},t) = \Sigma\, \tilde{u}_i(\mathbf{k},t)\exp(i\mathbf{k}.\mathbf{x}), \quad b_{ij}(\mathbf{x},t) = \Sigma\, \tilde{b}_{ij}(\mathbf{k},t)\exp(i\mathbf{k}.\mathbf{x}), \tag{2.4}$$

with similar relations for the generalized fields. The sums in (2.4) are over all $\mathbf{k}$ admitted by the boundary conditions. In what follows, we shall omit the tilde where there is no danger of confusing $\mathbf{x}$-space and $\mathbf{k}$-space quantities.

Homogeneity implies that amplitudes associated with distinct wave vectors are uncorrelated while isotropy and incompressibility imply that the covariance of the velocity field has the form

$$(L/2\pi)^D \langle u_i(\mathbf{k},t)\, u_j^*(\mathbf{k},t')\rangle = (D-1)^{-1} P_{ij}(\mathbf{k})\, U(k,t,t'), \tag{2.5}$$

where D is the dimensionality and

$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2. \tag{2.6}$$

The normalization in (2.5) gives

$$\langle |\mathbf{u}(\mathbf{x},t)|^2\rangle = \int U(k,t,t)\, d\mathbf{k} \tag{2.7}$$

when $L \to \infty$. Since the field $\mathbf{u}(\mathbf{x},t|s)$ need not be incompressible for $t \neq s$, the covariance of its transform has the more general form

$$(L/2\pi)^D \langle u_i(\mathbf{k},t|s)\, u_j^*(\mathbf{k},t'|s')\rangle = (D-1)^{-1} P_{ij}(\mathbf{k})\, U(k,t|s,t'|s') + k^{-2}k_i k_j\, V(k,t|s,t'|s'). \tag{2.8}$$

But incompressibility of the Eulerian field requires that the scalar V vanishes if $t = s$ or $t' = s'$.

By (1.1) and (2.5),

$$(L/2\pi)^D \langle b_{ij}(\mathbf{k},t)\, b_{mn}^*(\mathbf{k},t')\rangle = (D-1)^{-1} Q_{ijmn}(\mathbf{k})\, U(k,t,t'), \tag{2.9}$$

where
$$Q_{ijmn}(\mathbf{k}) = k_j k_n P_{im}(\mathbf{k}) + k_j k_m P_{in}(\mathbf{k}) + k_i k_n + P_{jm}(\mathbf{k}) k_i k_m P_{jn}(\mathbf{k}). \tag{2.10}$$

The most general isotropic reflexion-invariant tensor $T_{ijmn}(\mathbf{k})$ which is symmetric in i, j and in m, n can be expressed in terms of six independent scalars. But because of (2.2), the covariance of the generalized strain field $b_{ij}(\mathbf{k},t|s)$ depends on only three independent scalars. These may be chosen in an infinity of ways. It is convenient here to take the form

$$(L/2\pi)^D \langle b_{ij}(\mathbf{k},t|s)\, b_{mn}^*(\mathbf{k},t'|s')\rangle = (D-1)^{-1} Q_{ijmn}(\mathbf{k})\, U^B(k,t|s,t'|s') + Q'_{ijmn}(\mathbf{k})\, V^B(k,t|s,t'|s') + Q''_{ijmn}(\mathbf{k})\, W^B(k,t|s,t'|s'), \tag{2.11}$$

where $Q'_{ijmn}(\mathbf{k})$ and $Q''_{ijmn}(\mathbf{k})$ are any linear combinations of $Q_{ijmn}(\mathbf{k})$ with

$$k^2[\delta_{ij}\delta_{mn} - \tfrac{1}{2}D(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})], \quad k^2(\delta_{ij} - Dk_i k_j/k^2)(\delta_{mn} - Dk_m k_n/k^2) \tag{2.12}$$

which yield the orthogonality relations

$$Q_{ijmn}(\mathbf{k})\, Q'_{ijmn}(\mathbf{k}) = Q'_{ijmn}(\mathbf{k})\, Q''_{ijmn}(\mathbf{k}) = Q''_{ijmn}(\mathbf{k})\, Q_{ijmn}(\mathbf{k}) = 0. \tag{2.13}$$

Note from the form of (2.12) that Q' and Q'', as well as Q, satisfy relations of the form

$$Q_{ijmn}(\mathbf{k}) = Q_{jimn}(\mathbf{k}) = Q_{mnij}(\mathbf{k}), \quad Q_{iimn}(\mathbf{k}) = 0. \tag{2.14}$$

In all of the preceding we have assumed mirror symmetry. Without this assumption the general form of the isotropic strain covariance is quite complicated.

Application of the Eulerian specialization $t = s$ and $t' = s'$ to (2.11) gives back (2.9), so that

$$U^B(k,t|t,t'|t') = U(k,t|t,t'|t'), \quad V^B(k,t|t,t'|t') = W^B(k,t|t,t'|t') = 0. \tag{2.15}$$

The purely Lagrangian specialization $t = s$ and $t' = t$ will be of particular interest to us. The most general form of the Eulerian velocity–Lagrangian strain covariance in the isotropic mirror-symmetric case is

$$(L/2\pi)^D \langle u_i(\mathbf{k},t)\, b^*_{mn}(\mathbf{k},t|s)\rangle = (D-1)^{-1}(-i)\,[k_n P_{im}(\mathbf{k}) + k_m P_{in}(\mathbf{k})]\, U^B(k,t|t,t|s), \tag{2.16}$$

where we have used the solenoidal property with respect to i, the symmetry in m and n, and (2.2) to restrict the possible terms. Using

$$b_{ij}(\mathbf{k},t) = ik_i u_j(\mathbf{k},t) + ik_j u_i(\mathbf{k},t),$$

we find

$$(L/2\pi)^D \langle b_{ij}(\mathbf{k},t)\, b^*_{mn}(\mathbf{k},t|s)\rangle = (D-1)^{-1} Q_{ijmn}(\mathbf{k})\, U^B(k,t|t,t|s), \tag{2.17}$$

which shows that U^B is the same function in (2.16) as in (2.11) and that

$$V^B(k,t|t,t|s) = W^B(k,t|t,t|s) = 0. \tag{2.18}$$

In addition to the covariance functions, the renormalized perturbation analysis uses Green's functions which describe the averaged response to infinitesimal perturbations propagated according to the equations of motion of the Eulerian and generalized fields. The Green's function tensor for the velocity field has been discussed in detail (Kraichnan 1965). In the reflexion-invariant isotropic case it has the form

$$G_{ij}(\mathbf{k},t|s,t'|s') = P_{ij}(\mathbf{k})\, G(k,t|s,t'|s') + k^{-2}k_i k_j G'(k,t|s,t'|s'), \tag{2.19}$$

where $G'(k,t|s,t'|s')$ vanishes if $t = s$ or $t' = s'$. Note that the factor $(D-1)^{-1}$ in (2.8) is omitted in (2.19). The latter equation is normalized such that

$$G(k,t|s,t|s) = 1. \tag{2.20}$$

We reserve discussion of the initial condition ($t = t'$, $s = s'$) on $G'(k,t|s,t'|s')$ until §4. In correspondence to the covariance behaviour, $G'(k,t|t,t|s)$ vanishes, as do

$$G'(k,t|t,t'|s') \quad \text{and} \quad G'(k,t|s,t'|t').$$

We shall denote the average infinitesimal Green's tensor of the generalized strain field by $G^B_{ijmn}(k,t|s,t'|s')$. It describes the propagation of infinitesimal perturbations of the strain field through the t, s plane according to (1.5) and the Navier–Stokes equation, along the same integration path as was described for the velocity-field Green's tensor in Kraichnan (1965). The general isotropic mirror-symmetric form for G^B_{ijmn} is

$$G^B_{ijmn}(\mathbf{k}, t|s, t'|s') = Q_{ijmn}(\mathbf{k})G^B(k, t|s, t'|s') + \text{terms in } Q'_{ijmn}(\mathbf{k}) \text{ and } Q''_{ijmn}(\mathbf{k}), \quad (2.21)$$

with
$$G^B(k, t|s, t|s) = 1. \quad (2.22)$$

The terms in Q' and Q'' vanish if $t = s$, $t' = s'$ or $t = s = t'$, while

$$G^B(k, t|t, t'|t') = G(k, t|t, t'|t'). \quad (2.23)$$

3. Strain covariance for a frozen velocity field

The method given in I for constructing the Lagrangian-history expansions may be summarized as follows. First introduce the solution $u^0_i(\mathbf{x}, t)$ of the linearized Navier–Stokes equation (nonlinear terms removed) and the corresponding field $u^0_i(\mathbf{x}, t|s)$ which satisfies (1.4) with the nonlinear term removed. Equation (1.4) is then degenerate and yields simply

$$u^0_i(\mathbf{x}, t|s) = u^0_i(\mathbf{x}, s). \quad (3.1)$$

Assume (for our present purpose) that $u^0_i(\mathbf{x}, t)$ has a multivariate-Gaussian isotropic distribution. Now reintroduce the nonlinear terms in the equations of motion as perturbations, solve by iteration and obtain $u_i(\mathbf{x}, t|s)$ as a functional power series in $u^0_i(\mathbf{x}, t|s)$. Use these expansions, together with averaging over the $\mathbf{u}^0$ distribution, to express moments of $\mathbf{u}$, in particular triple moments and the covariance scalar $U(k, t|s, t'|s')$, as functional power series in the covariance scalar $U^0(k, t|s, t'|s')$. The crucial steps then comprise reverting the series for U to obtain U^0 as a functional power series in U and substituting the latter series into the expansions for the triple moments to express, finally, the latter as power series in U. This procedure yields the Eulerian renormalized expansion, the LH expansion or the ALH expansion, the choice being made by exploiting the independence of $u^0_i(x, t|s)$ of the measuring time t when reverting the covariance series. In the Eulerian case, only Eulerian moments with $t = s$ and $t' = s'$ need be considered. In the LH case, a closed system is formed from moments of the form $U(k, t|s, t|s')$ by using (3.1) to alter all U^0 functions to that form before reverting the series for U. In the ALH case, a similar procedure is followed but the closed set involves only moments of the form $U(k, t|t, t|s)$. Further details are given in I.

In the present analysis we instead introduce the linearized straining field $b^0_{ij}(\mathbf{k}, t|s)$ in a similar fashion and exploit the relation

$$b^0_{ij}(\mathbf{k}, t|s) = b^0_{ij}(\mathbf{k}, s). \quad (3.2)$$

From (1.1), (3.1) and (3.2) we have

$$U^{B0}(k, t|s, t'|s') = U^0(k, t|s, t'|s'). \quad (3.3)$$

We may then express $U^B(k, t|s, t'|s')$ as a functional power series in $U^0(k, t|s, t'|s')$, revert the series after the desired alteration of labelling times, and use the result to express triple moments as functional series in U^B. In particular, the single-time triple moments which control energy transfer may be so expressed. In the Eulerian case, the final expansion for triple moments is the same as in the original velocity-based analysis, because of (1.1). But inequivalent results are obtained in the LH and ALH cases, in consequence of (1.6). The lowest truncations of the strain-based LH and ALH expansions yield the SBLHDI and SBALHDI approximations.

We shall start by treating a frozen velocity field $\mathbf{u}(\mathbf{x}, t)$ constant in time, so that the only equation of motion is (1.4) or (1.5). This of course greatly simplifies the analysis. A bonus is that only Green's functions with time arguments of the form $(t|s, t|s)$ appear in the expansions, after alteration of labelling times in the zeroth-order functions, so that by (2.20) and (2.22) all relevant Green's-function scalars are identically equal to one. This means that Green's functions need not be explicitly introduced at all.

The LH and ALH analysis for the Lagrangian velocity covariance $U(k, t|t, t|s)$ was given in I, using an $\mathbf{x}$-space representation. It is easier to carry out the SBLH and SBALH analysis in $\mathbf{k}$ space. We start by iterating

$$\partial b_{ij}(\mathbf{k}, t|s)/\partial t = -ik_m \Sigma_\Delta u_m(\mathbf{q}, t)\, b_{ij}(\mathbf{p}, t|s), \tag{3.4}$$

which is the transform of (1.5). Here Σ_Δ denotes a sum over all $\mathbf{p}$ and $\mathbf{q}$ such that $\mathbf{k} = \mathbf{p}+\mathbf{q}$. The result is

$$b_{ij}(\mathbf{k}, t|s) = b^0_{ij}(\mathbf{k}, t|s) - ik_m \Sigma_\Delta \int_0^t u^0_m(\mathbf{q}, s'|s')\, b^0_{ij}(\mathbf{p}, s'|s)\, ds' + \text{higher-order terms}, \tag{3.5}$$

where we have used $\mathbf{u}(\mathbf{q}, s') = \mathbf{u}(\mathbf{q}, s'|s')$. Multiplication of (3.5) by its complex conjugate and averaging over the Gaussian distribution of the zeroth-order fields proceeds as in I. Then by using the projection operator $Q_{ijmn}(\mathbf{k})$ we obtain the scalar expansion

$$U^B(k, t|s, t|s') = U^0(k, t|s, t|s') + \text{quadratic and higher-order terms in } U^0. \tag{3.6}$$

Only the lowest-order term, shown explicitly, and only the case $t' = t$ which we have taken are needed to construct the SBLHDI and SBALHDI approximations.

Next all the intermediate (integrated-over) labelling times in (3.6) are changed to t according to the LH prescription of I. Reversion then gives

$$U^0(k, t|s, t|s') = U^B(k, t|s, t|s') + \text{quadratic and higher-order terms in } U^B. \tag{3.7}$$

An equation of motion for $U^B(k, t|s) \equiv U^B(k, t|t, t|s)$ is obtained by multiplying (3.4) by $-ik^{-2}k_j u^*_i(k, t)$ and averaging. This gives

$$\partial U^B(k, t|s)/\partial t = S^B(k, t|s), \tag{3.8}$$

where
$$S^B(k, t|s) = -i(L/2\pi)^D k_m \Sigma_\Delta \langle u_m(\mathbf{q}, t)\, b_{ij}(\mathbf{p}, t|s)\, u^*_i(\mathbf{k}, t)\rangle, \tag{3.9}$$

and we note from (2.16) and (2.6) that

$$-i(L/2\pi)^D k^{-2} k_j \langle u^*_i(\mathbf{k}, t)\, b_{ij}(\mathbf{k}, t|s)\rangle = U^B(k, t|t, t|s). \tag{3.10}$$

We should also note here that U and U^B are real scalars and that

$$u^*_i(\mathbf{k}, t|s) = u_i(-\mathbf{k}, t|s), \quad b^*_{ij}(\mathbf{k}, t|s) = b_{ij}(-\mathbf{k}, t|s),$$

since the fields in $\mathbf{x}$ space are real.

Now we substitute (3.5) into the right-hand side of (3.9) and average to obtain

$$\begin{aligned} S^B(k, t|s) = -i(L/2\pi)^D k_j k_m \Sigma_\Delta \int_s^t p_n[&\langle u^0_m(\mathbf{q}, t)\, u^{0*}_n(\mathbf{q}, s'|s')\rangle \langle b^0_{ij}(\mathbf{k}, t|s)\, u^{0*}_i(\mathbf{k}, t)\rangle \\ &+ \langle u^0_m(\mathbf{q}, t)\, b^0_{ij}(\mathbf{q}, s'|s)\rangle \langle u^0_n(\mathbf{k}, s'|s')\, u^{0*}_i(\mathbf{k}, t)\rangle]\, ds' + \text{higher-order terms}. \end{aligned} \tag{3.11}$$

Here we have used the multivariate-normal property of $\mathbf{u}^0$, the fact that distinct wave vectors are uncorrelated because of homogeneity, and the fact that

$$\mathbf{u}(\mathbf{k},t) = \mathbf{u}^0(\mathbf{k},t) = \mathbf{u}^0(\mathbf{k},t|t)$$

because the field is frozen. Then the intermediate labelling times s', etc. are changed to t, the covariances are reduced to scalars by the isotropic formulae and the zeroth-order scalars are replaced by expansions in U^B according to (3.7). The result is

$$\partial U^B(k,t|s)/\partial t = (2\pi/L)^D \Sigma_\Delta \int_s^t ds'[C_0 U^B(k,t|s')\, U^B(q,t|s) - C_1 U^B(k,t|s)\, U^B(q,t|s')], \text{ + higher-order terms,} \quad (3.12)$$

where
$$C_0 = (D-1)^{-2} k^{-2} k_j k_m p_n P_{ni}(\mathbf{k})\,[q_i P_{jm}(\mathbf{q}) + q_j P_{im}(\mathbf{q})]$$
$$= -(D-1)^{-2} k^{-2} k_j k_m q_n P_{ni}(\mathbf{k})\,[q_i P_{jm}(\mathbf{q}) + q_j P_{im}(\mathbf{q})], \quad (3.13)$$
$$C_1 = (D-1)^{-2} k^{-2} k_j k_m p_n P_{mn}(\mathbf{q})\,[k_i P_{ij}(\mathbf{k}) + k_j P_{ii}(\mathbf{k})]$$
$$= (D-1)^{-1} k_m k_n P_{mn}(\mathbf{q}) \quad (3.14)$$

and we have used

$$\mathbf{k} = \mathbf{p} + \mathbf{q}, \quad k_i P_{ij}(\mathbf{k}) = q_n P_{mn}(\mathbf{q}) = 0, \quad P_{ii}(\mathbf{k}) = D-1.$$

Equation (3.12) may be simplified by using

$$(2\pi/L)^D \Sigma_\Delta \to \int d\mathbf{q} \quad (L \to \infty) \quad (3.15)$$

and the isotropic angle-average formulae

$$\left.\begin{aligned} (D-1)^{-1}\{P_{ij}(\mathbf{q})\} &= D^{-1}\delta_{ij}, \\ \{q_n q_i P_{jm}(\mathbf{q})\} &= q^2[D^{-1}\delta_{ni}\delta_{jm} - (D(D+2))^{-1}(\delta_{ni}\delta_{jm} + \delta_{mi}\delta_{nj} + \delta_{mn}\delta_{ij})] \end{aligned}\right\} \quad (3.16)$$

to yield (retaining only lowest-order terms)

$$\partial U^B(k,t|s)/\partial t = -\int d\mathbf{q} \int_s^t ds' \{D^{-1} k^2 U^B(k,t|s) U^B(q,t|s') + [(D-1)(D+2)]^{-1} q^2 U^B(q,t|s)\, U^B(k,t|s')\}. \quad (3.17)$$

If the above analysis is performed instead by proceeding from the transform of (1.4) and expressing triple moments as series in U instead of U^B, we obtain (to lowest order)

$$\partial U(k,t|s)/\partial t = (2\pi/L)^D \Sigma_\Delta \int_s^t ds'[C_0' U(k,t|s')\, U(q,t|s) - C_1 U(k,t|s)\, U(q,t|s')], \quad (3.18)$$

where
$$C_0' = (D-1)^{-2} q_n P_{ni}(\mathbf{k})\, P_{im}(\mathbf{q})\, k_m \quad (3.19)$$

and C_1 is still given by (3.14). The isotropic angle-average of C_0' is zero, so that (3.18), the LHDI approximation for the Lagrangian velocity covariance, takes the final form

$$\partial U(k,t|s)/\partial t = -D^{-1} k^2 U(k,t|s) \int d\mathbf{q} \int_s^t ds' U(q,t|s'). \quad (3.20)$$

This is just the transform of the $\mathbf{x}$-space result obtained in I.

There are several interesting contrasts between (3.17) and (3.20). First, the initial curvature of $U^B(k,t|s)$ exceeds that of $U(k,t|s)$ if both are considered as functions of $t-s$. Since the truncations of the perturbation series are exact at second order, these initial curvatures are also exact. In general this suggests that the Lagrangian straining covariance exhibits a shorter correlation time than the Lagrangian velocity covariance. If the Eulerian spectrum is a δ-function in k, (3.17) and (3.20) are solvable analytically, as described in I. The ratio of the correlation times is then a monotonic function of D ($D \geqslant 2$), rising to the value one at $D = \infty$, where (3.17) and (3.20) are identical. For $D = 3$ the ratio is $(\frac{10}{13})^{\frac{1}{2}}$ and for $D = 2$ it is $(\frac{2}{3})^{\frac{1}{2}}$.

A peculiar feature of (3.17) is that the initial curvature does not go to zero with k, in contrast to (3.20). This reflects the different effects of small-scale turbulence on large scales of the straining field as opposed to the velocity field. The effect on the velocity field is an eddy-viscosity type of diffusion. But fluid elements can be substantially rotated in a time of the order of the eddy circulation time of the small scales, thereby decorrelating the rate of strain in the fluid element with its initial value even though the distance of migration through the large-scale field is small.

Equations (3.12) and (3.18) are the lowest truncations of the SBALH and ALH expansions, respectively, as well as the lowest truncations of the SBLH and LH expansions. As explained in I, the higher terms of the expansions are not equivalent; closed sets involving only time arguments of the form $(t|t,t|s)$ are obtained from the SBALH and ALH expansions, but the closed sets resulting from the SBLH and LH expansions involve functions with more general time arguments $(t|s,t|s')$. In the case of the Navier–Stokes equation, closed sets involving only $(t|t,t|s)$ type arguments arise only from the SBALH and ALH expansions even at the lowest level of truncation.

4. Navier–Stokes turbulence

The construction of the SBALH and SBLH expansions for the Navier–Stokes system follows the same logic as for the frozen-field example, but with some added complications. First, the Green's functions do not all drop out trivially, and must be carried to the end. Second, we must handle the fact that the linearized covariances involve only a single scalar while, according to (2.11), the actual covariance can involve three scalars. Similarly, the actual Green's function involves three scalars. This implies an *a priori* ambiguity in reverting the expansions of actual covariances and Green's functions in terms of linearized covariances and Green's functions. This ambiguity has already appeared in the frozen-field problem but we have passed over it in order to consolidate the discussion here.

In the original formulation of the LHDI approximation (Kraichnan 1965), the treatment of the Green's functions was simplified by introducing a fictitious compressive part of the Eulerian velocity which simply decayed under viscosity and did not advect. This led to the simple initial condition

$$G_{ij}(\mathbf{x},t|s;\ \mathbf{x}',t|s) = \delta_{ij}\delta(\mathbf{x}-\mathbf{x}'). \tag{4.1}$$

The scalars U and V in (2.8) and G and G' in (2.19) were then treated on an equal footing. This has certain advantages but it leads to cumbersome expansions, particularly in the present strain-based analysis. The compressive Eulerian velocity gives a straining-field contribution which does not obey (2.1 a), with the result that even more scalars must be admitted in (2.11).

In the present work we instead wish to deal only with solenoidal tensors in the reverted expansions, as in I. To this end we suppose that the fictitious compressive Eulerian velocity encounters an infinite viscosity so that any perturbations which originate off the diagonal of the t, s plane become solenoidal on the diagonal. This is equivalent to requiring that V in (2.8) vanishes if $t = s$ or $t' = s'$ and that G' in (2.19) obeys the initial conditions

$$G'(k,t|s,t|s) = 1 \quad (t \neq s), \quad G'(k,t|t,t|t) = 0 \tag{4.2}$$

with corresponding conditions on the Q' and Q'' terms in (2.21). The linearized Green's functions are then purely solenoidal and the necessary reversions are solely those of the expansions of the scalars U and G in powers of U^0 and G^0 in the velocity-based case and of U^B and G^B in the strain-based case. These expansions can be isolated by use of the projection operators P and Q.

With this background the SBLH and SBALH expansions for triple moments and for the moments which give the time derivatives of the Green's function are determined by techniques given in I and our discussion of the frozen-field case. The tediousness of the analysis is not really forbidding if attention is restricted to the SBALHDI approximation, i.e. the lowest terms of the SBALH expansion. Then great simplifications result from judicious use of (2.16). The final form of the SBALHDI equations, reduced to scalars, is as follows:

$$(\partial/\partial t + 2\nu k^2)\, U^B(k,t|t) = 2\int d\mathbf{q} \int_0^t ds (D-1)^{-2} C_3 [G^B(k,t|s)\, U^B(p,t|s) - G^B(p,t|s)\, U^B(k,t|s)]\, U^B(q,t|s), \tag{4.3}$$

$$\begin{aligned}(\partial/\partial t + \nu k^2)\, U^B(k,t/s) &= (D-1)^{-2} \int d\mathbf{q} \Big\{ C_0 \int_s^t U^B(k,t|s')\, U^B(q,t|s)\, ds' - (D-1)\, C_1 U^B(k,t|s) \\ &\times \int_s^t U^B(q,t|s')\, ds' + C_2 U^B(p,t|s) \int_s^t U^B(q,t|s')\, ds' \\ &+ \int_0^s [C_3 G^B(k,s|s')\, U^B(p,t|s') - C_4 G^B(p,s|s')\, U^B(k,t|s')]\, U^B(q,t|s')\, ds' \\ &- \int_0^t [C_5 G^B(p,t|s')\, U^B(k,s|s') - C_6 G^B(k,t|s')\, U^B(p,s|s')]\, U^B(q,t|s')\, ds' \\ &- \int_0^t C_7 G^B(p,t|s')\, U^B(k,t|s')\, U^B(q,s|s')\, ds' \Big\},\end{aligned} \tag{4.4}$$

$$\begin{aligned}(\partial/\partial t + \nu k^2)\, G^B(k,t|s) &= (D-1)^{-2} \int d\mathbf{q} \Big\{ C_0 \int_s^t G^B(k,t|s')\, U^B(q,t|s)\, ds' - (D-1)\, C_1 G^B(k,t|s) \\ &\times \int_s^t U^B(q,t|s')\, ds' + C_2 G^B(p,t|s) \int_s^t U^B(q,t|s')\, ds' \\ &- \int_s^t [C_3 G^B(p,t|s')\, G^B(k,s'|s) - C_4 G^B(p,s'|s)\, G^B(k,t|s')]\, U^B(q,t|s')\, ds' \\ &- C_8 \int_s^t G^B(p,t|s)\, U^B(q,t|s)\, G^B(k,s'|s)\, ds' \Big\}.\end{aligned} \tag{4.5}$$

Here $\mathbf{p}+\mathbf{q} = \mathbf{k}$ and the coefficients are given by

$$C_0 = \mathbf{k}.\mathbf{P}'(-\mathbf{q}).\mathbf{P}(\mathbf{k}).\mathbf{q}, \quad C_1 = \mathbf{k}.\mathbf{P}(\mathbf{q}).\mathbf{k}, \tag{4.6a,b}$$

$$C_2 = \mathbf{k}.\mathbf{P}'(\mathbf{p}).\mathbf{P}(\mathbf{k}).\mathbf{P}(\mathbf{q}).\mathbf{k}+\mathbf{P}(\mathbf{k}):\mathbf{P}'(\mathbf{p})[\mathbf{k}.\mathbf{P}(\mathbf{q}).\mathbf{k}], \tag{4.6c}$$

$$C_3 = \mathbf{k}.\mathbf{P}(\mathbf{p}).\mathbf{P}(\mathbf{k}).\mathbf{P}(\mathbf{q}).\mathbf{p}+\mathbf{k}.\mathbf{P}(\mathbf{p}).\mathbf{P}(\mathbf{q}).\mathbf{P}(\mathbf{k}).\mathbf{p}+\mathbf{k}.\mathbf{P}(\mathbf{q}).\mathbf{P}(\mathbf{p}).\mathbf{P}(\mathbf{k}).\mathbf{p} + \mathbf{P}(\mathbf{k}):\mathbf{P}(\mathbf{p})[\mathbf{k}.\mathbf{P}(\mathbf{q}).\mathbf{k}], \tag{4.6d}$$

$$C_4 = \mathbf{p}.\mathbf{P}(\mathbf{q}).\mathbf{P}'(\mathbf{p}).\mathbf{P}(\mathbf{k}).\mathbf{p}+\mathbf{P}(\mathbf{k}):\mathbf{P}'(\mathbf{p})[\mathbf{k}.\mathbf{P}(\mathbf{q}).\mathbf{k}], \quad C_5 = C_3, \quad C_6 = C_2, \tag{4.6e–g}$$

$$C_7 = \mathbf{k}.\mathbf{P}(\mathbf{p}).\mathbf{P}'(\mathbf{q}).\mathbf{P}(\mathbf{k}).\mathbf{p}+\mathbf{p}.\mathbf{P}'(\mathbf{q}).\mathbf{P}(\mathbf{k}).\mathbf{P}(\mathbf{p}).\mathbf{k}, \tag{4.6h}$$

$$C_8 = \mathbf{k}.\mathbf{P}''(-\mathbf{q}).\mathbf{P}(\mathbf{p}).\mathbf{P}(\mathbf{k}).\mathbf{p}+\mathbf{p}.\mathbf{P}(\mathbf{k}).\mathbf{P}''(-\mathbf{q}).\mathbf{P}(\mathbf{p}).\mathbf{k}, \tag{4.6i}$$

where $\mathbf{P}(\mathbf{k}):\mathbf{P}(\mathbf{p})[\mathbf{k}.\mathbf{P}(\mathbf{q}).\mathbf{k}] = P_{ij}(\mathbf{k})\,P_{ij}(\mathbf{p})\,k_n\,P_{nm}(\mathbf{q})\,k_m,$ etc.,

and

$$\mathbf{P}'(\mathbf{p}) = k^{-2}[(\mathbf{k}.\mathbf{p})\,\mathbf{P}(\mathbf{p})+(\mathbf{P}(\mathbf{p}).\mathbf{k})\,\mathbf{p}],$$

$$\mathbf{P}''(\mathbf{q}) = p^{-2}[(\mathbf{p}.\mathbf{q})\,\mathbf{P}(\mathbf{q})+(\mathbf{P}(\mathbf{q}).\mathbf{p})\,\mathbf{q}]. \tag{4.7}$$

The initial time when the velocity field is multivariate normal is $t = 0$. The initial condition on $G^B(k,t|s)$ is

$$G^B(k,s|s) = 1. \tag{4.8}$$

The Eulerian modal intensity is given according to (2.17) by

$$U(k,t,t) = U^B(k,t|t), \tag{4.9}$$

and for $t = 0$ this gives the initial condition on $U^B(k,t|s)$.

The integration over $\mathbf{q}$ in (4.3)–(4.5) may be expressed as an integration over scalars p and q according to (Fournier & Frisch 1978)

$$\int d\mathbf{q} = \iint_\Delta A_{D-1}(pq/k)^{D-2}\,(\sin\alpha)^{D-3}\,dp\,dq, \tag{4.10}$$

where

$$A_D = 2\pi^{\frac{1}{2}D}/\Gamma(\tfrac{1}{2}D) \tag{4.11}$$

is the $(D-1)$-dimensional surface area of a unit sphere in D dimensions, $\iint_\Delta$ denotes integration over the strip in the p,q plane such that k,p and q can form a triangle, and α is the interior angle opposite k in that triangle.

The original velocity-based ALHDI approximation of Kraichnan (1965) is recovered from (4.3)–(4.11) by simply replacing all U^B and G^B functions by U and G functions with the same arguments and replacing all P' and P'' functions by P functions with the same arguments. The coefficients given in (4.6) differ in form from those of Kraichnan (1965) because they have not been reduced to explicit functions of the parts of the k, p, q triangle. Note that, after the replacement of the $\mathbf{P}'(-\mathbf{q})$ by $\mathbf{P}(-\mathbf{q})$, C_0 gives zero on averaging over angles. Apart from a different normalization with $D-1$, C_0 and C_1 are the same coefficients as were defined in (3.13) and (3.14). The C_0 and C_1 terms in (4.4) are identical with those in the frozen-field equation (3.12).

5. Discussion

The strain-based expansions and the SBALHDI approximation to which they lead are motivated by the quantitative inaccuracies of the LHDI and ALHDI approximations described in § 1. Having now exhibited the SBALHDI equations for isotropic turbulence, can we offer any reason for feeling that they may in fact be an improvement? The analysis of the frozen field in § 3 showed a somewhat smaller correlation time for the Lagrangian straining covariance than for the Lagrangian velocity covariance. Since the Lagrangian correlation time helps to determine the magnitude of energy transfer in (4.3) and the frozen-field terms reappear in (4.4), this effect is in the correct direction to suggest a modest decrease in energy transfer in the SBALHDI approximation. Potentially much more important improvement is associated with the dissipation-range behaviour in the Navier–Stokes case. Consider the Lagrangian covariance

$$B(t|s) = \langle b_{ij}(\mathbf{x},t)\, b_{ij}(\mathbf{x},t|s)\rangle = 2\int k^2 U^B(k,t|s)\, d\mathbf{k}. \tag{5.1}$$

It can be shown from (4.3) that

$$\int_0^t B(t|s)\, ds$$

is a measure of the energy transfer rate in very small scales: wavenumbers high in the dissipation range. In the ALHDI approximation, the corresponding role is played by

$$B'(t|s) = \langle b_{ij}(\mathbf{x},t)\, b'_{ij}(\mathbf{x},t|s)\rangle = 2\int k^2 U(k,t|s)\, dk, \tag{5.2}$$

where $b'_{ij}(\mathbf{x},t|s)$ is the right-hand side of (1.6). If the turbulence is stationary (or quasi-stationary in the small scales), the Lagrangian straining is also stationary, with the result that $B(t|s)$ is an even function of $t-s$. In particular, its slope vanishes at $t = s$. On the other hand it is easy to show (Kraichnan 1966*a*) that

$$[\partial U(k,t|s)/\partial t]_{t=s} = S(k,t), \tag{5.3}$$

where $2S(k,t)$ is the contribution of the nonlinear terms to $\partial U(k,t|t)/\partial t$. Equation (5.3) is an exact relation. In the ALHDI approximation $2S(k,t)$ is the right-hand side of (4.3), with U^B and G^B replaced by U and G. Equation (5.3) gives

$$[\partial B'(t|s)/\partial t]_{t=s} = 2\int k^2 S(k,t)\, d\mathbf{k}. \tag{5.4}$$

The right-hand side of (5.4) measures the rate of enstrophy production and in general is positive, $D = 2$ excepted. Thus $B'(t|s)$ starts off with a positive slope, in contrast to the zero initial slope of $B(t|s)$. This suggests that B' has the longer correlation time and is associated with stronger energy transfer at small scales. We have then additional evidence that the SBALHDI approximation gives lower energy transfer than the ALHDI approximation, with an indication that the difference shows up most strongly in the dissipation range.

The quadratic constants of the inviscid Navier–Stokes equation survive at each order of the LH and ALH expansions. The argument in I which establishes this applies equally to the SBLH and SBALH expansions. In the ALHDI and SBALHDI approximations the quadratic conservation properties can be verified directly from the form of the coefficient C_3 which appears in (4.3).

Matters become more complicated when the equipartition–equilibrium and fluctuation–dissipation relations of the inviscid equations truncated at a cut-off wavenumber are considered. In I it was noted that the exact equipartition–equilibrium relation

$$U(k,t|s,t'|s') \propto G(k,t|s,t'|s') \quad (s \geqslant s') \tag{5.5}$$

survives at each order of the ALH and LH expansions only if $t' = t$. (For the ALH expansion the relation is defined only if also $s = t$). However

$$U^B(k,t|s,t'|s') \propto G^B(k,t|s,t'|s') \quad (s \geqslant s') \tag{5.6}$$

is not an exact relation for an equilibrium of the inviscid truncated Navier–Stokes system. The difference arises because both the Navier–Stokes equation and (1.4) exhibit quadratic constants in the form of sums of squares of **u** amplitudes, while (1.5) does not have such a constant. That is, (1.4) gives

$$\partial\left[\int u_i(\mathbf{x},t|s)\,u_i(\mathbf{x},t|s')\,d\mathbf{x}\right]\Big/\partial t = 0 \tag{5.7}$$

while (1.5) gives

$$\partial\left[\int b_{ij}(\mathbf{x},t|s)\,b_{ij}(\mathbf{x},t|s')\,d\mathbf{x}\right]\Big/\partial t = 0. \tag{5.8}$$

The consequence is that the method of Kraichnan (1965, appendix A) which establishes (5.5) does not apply to the case where (1.5) replaces (1.4). A further consequence is that the intimate mixing of (1.5) and the Navier–Stokes equation effected by the SBLH and SBALH reversions destroys the energy-equipartition property for truncations of the SBLH and SBALH expansions.

In the SBALHDI approximation, the destruction of the energy-equipartition solution shows up only from the action of the C_8 term in (4.5). If that coefficient only is replaced by the ALHDI value, the energy-equipartition solution is restored.

In two dimensions, where enstrophy is a constant of the inviscid Navier–Stokes equation, the situation is reversed. Enstrophy equipartition is also straining equipartition and

$$k^2U^B(k,t|s,t'|s') \propto G^B(k,t|s,t'|s') \quad (s \geqslant s') \tag{5.9}$$

is exact for that equilibrium. This survives for $t = t'$ in the strain-based expansions and the SBALHDI approximation has an enstrophy-equipartition solution. The ALHDI approximation does not have such a solution because (1.4) has no quadratic constants related to enstrophy.

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